Is Japanese music more consonant than Western music?
an application of Leonhard EULER's music theory

日本音楽は西洋音楽より協和的なのか

— レオンハルト・オイラーの音楽論の適用 —

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1. Introduction

In his last lecture given at Ochanomizu University, Tokumaru Yosihiko mentions that *syamisen* (*shamisen*) players tend to play the second note of the scale a half tone above the key note in the lower octave, but a whole tone in the upper octave (ex. 1a). This seems to contradict the rule that musical scales repeat their tones in each octave. Tokumaru shows, however, that this "irregularity" conforms to a tonal system based on tetrachords, or, as he calls them, latent units (潜在単位), rather than on octaves. The structure of this scale also reflects the structure of the instrument and its playing techniques (Tokumaru 2003: 16 - 19, 32).

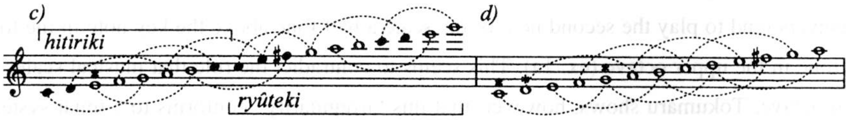
But this kind of scale is not exclusive to *syamisen* music. *Kumoi dyôsi* (*kumoi-jôshi*) for instance, one of the fundamental *koto* tunings, shows the same difference between the third/eighth and the last string (ex. 1b). In *gagaku* a similar scale can be observed, if the pitches of the *hitiriki* (*hichiriki*) and the *ryûteki* are combined (ex. 1c). Significantly enough, the two instruments, which essentially play variations of the same melody one octave apart from each other, use the differing tones, i.e., the dissonant interval of an augmented octave, even simultaneously.

The phenomenon is also not restricted to Japan. In 13th to 16th century European music manuscripts, for example, frequently a *b*-flat is found in the key signature of the lower voice(s), but not in the upper voice(s)<1>. Thus the tone material of these pieces (ex. 1d) is almost the same as that of *syamisen* music, even if the keynote is at another place, and the tone progressions are very different from Japanese music.

NB. In example 1 the scales are transposed for comparative purposes: The original *gagaku* scale is a fifth higher and the original medieval notation a fifth lower than in this example. The most frequent notes in the *tôgaku* piece *Seigaiha* were calculated by a listening analysis of a recording by the *Kunaityô* (*Kunaichô*) *Gakubu* (1989), CD KICH 2001. "Most frequent" means used more than 25 times and "frequent" at least five times throughout the piece. The range of the medieval scale shows a typical case, but in some pieces more or fewer notes are used. The key note (*finalis*) of the medieval scale depends on the mode the piece is written in. In most pieces the finalis is *f* or *g* in original notation, i.e., *c* or *d* in the transposed scale.

The most frequently used pitches in syamisen music (hon-tyôsi) after Tokumaru. Whole notes show the "nuclear tones".

The koto-tuning kumoi-dyôsi.



Frequently used tones by the hitiriki and ryûteki in bansiki-tyô (banshiki-chô) Seigaiha. The most frequent tones are marked as whole notes.

A typical tone material of a European vocal piece of the 13th-16th century, written for three voices.

Ex. 1. Broken slurs indicate perfect fifths. The asterisk (*) indicates the key note (finalis) or, in the case of the medieval scale, possible key notes.

It is obvious that all of these scales came about under special conditions, including peculiarities of the instruments and playing techniques as well as the historical development of musical language. The explanation of these scales in music theoretical terms will accordingly be different for each of the scales. It may be asked, however, if the phenomenon in its most general sense — a diatonic or pentatonic scale that has a sharp (or natural) in the upper octave(s) where a natural (or flat) is found in the lower octave(s) — has a more general explanation.

It seems that the phenomenon has something to do with consonance between scale notes. In most cases the alteration increases the number of perfect fifths (see broken slurs in ex. 1). But why is an octave as the most consonant interval sacrificed for the sake of a fifth, which is less perfect? And why, if fifths are desirable, a *koto* tuning like *kumoi-dyôsi* is possible, which has no perfect fifths in the middle register? And why is the eleventh string in *hira-dyôsi* (the most common *koto* tuning, see below exx. 6 and 7) not sharpened as the thirteenth string is in *kumoi-dyôsi*?

To answer these questions, not only particular consonance relations, but the *consonance of the whole system* must be taken into consideration. Consonance, in general, refers both to tones that are played in succession and to those that are sounded together in a chord, and it means that they "fit together" in some way. Consonance of a whole system therefore refers to how much the tones are in an order which makes them all fit together. Thus it concerns sound possibilities and imagination rather than a specific sound effect. Although traditional music theory tends to consider only a few categories of consonance (e.g., "perfect consonance", "imperfect consonance", and "dissonance" in counterpoint theory), there exist many grades (and perhaps sorts) of "fitting together", and there is no definite borderline between consonant and dissonant tone relations.

It may be doubted whether or not consonance of a whole system is a one-dimensional unit of measurement. The mathematician Leonhard Euler (1707 - 1783), however, developed a very refined music theory<2> that attributes any tone system a grade of *suavitas* (sweetness) expressed by a natural number and closely related to the grade of consonance. Despite its restrictions (mainly because it is a pure mathematical theory that does not refer to the acoustical properties of real sound or the physiology of hearing), it reveals some effects of tone combinations that are hardly ever explained by any other music theory. In fact, the theory is also able to give a satisfying answer to our problem of scales.

2. Euler's music theory

Euler's music theory is based on only a few fundamental statements:

(1) *Suavitas* is formed from simple ratios between the frequencies of tones; thus an octave (frequency ratio 1 : 2) is more *suavis* than a fifth (frequency ratio 2 : 3) just because the ratio between the numbers 1 and 2 is simpler than that between 2 and 3. The grade of suavitas (*gradus suavitatis*, GS) is therefore a "grade of simplicity of ratio". Because simple things are easy to comprehend, it could also be called a "grade of comprehensibility". The GS is counted by natural numbers from the simplest, i.e., the unison (GS = 1), to the incomprehensible (GS = ∞) and can be calculated for any set of frequencies<3> that are in rational relation<4> to each other. (Note that for GS the first grade means the highest *suavitas* and a greater number means a lower *gradus suavitatis*!)

(2) Musicians like to create complex systems that are difficult to understand. Listeners like (perhaps unconsciously) to search for the order in such systems, and they are satisfied by finding it. Thus listening to music can be compared to solving a puzzle: If it is too easy or too difficult, the listener will be discontented. Good music, accordingly, must have a certain grade of difficulty that depends on the ability of the listener. (Euler says in the preface, p. 7, that this is the reason why we don't like the music of the "barbarians" and vice versa.)

(3) There are some methods to enable a listener to understand more complex structures. The structure, for example, can be presented more often. Thus

(*a*) tone relations between higher notes are easier to understand than between lower notes, because more vibrations affect the ear, and

(*b*) in slow movements more complex harmonies are possible than in quick movements, and lower voices need slower tone movements than higher voices (Euler 1739: II, 20).

More important is, however, to decompose complex structures into simpler units that are related to each other. If the units and their mutual relations are comprehended, the complex structure becomes comprehensible as a whole. This process goes through several levels: The tones in a chord, the relation between two chords, the chords in a phrase, the relation between two phrases, a group of phrases, and so on. The harmonies in a musical phrase, for example, constitute a musical key, and the keys used in a piece create the tone system the piece is based on. Through a skilled arrangement of harmonies and phrases, great composers are able to lead the listener to the comprehension of very complex structures (Euler 1739: VI, 12 - 15).

Scales, modes and tone systems can be regarded as steps in this process from simple to complex structures.

I will give a short outline of how from these fundamental statements a mathematical theory was formulated that enables us to evaluate the consonance and musical quality of a tone system.

The grade of simplicity, or *gradus suavitatis*, is first developed for single natural numbers. The simplest number is 1. Thus it is defined:

$$GS(1) = 1 \qquad [1]$$

Prime numbers cannot be further divided, so their simplicity depends only on their size. Thus it is defined:

$$\mathrm{GS}(p) = p, \text{ if } p \text{ is prime.} \quad [2]$$

A product mn is less simple than n in the same degree, as m is less simple than 1; i.e.,

$$\mathrm{GS}(mn) - \mathrm{GS}(n) = \mathrm{GS}(m) - \mathrm{GS}(1), \text{ or}$$

$$\mathrm{GS}(mn) = \mathrm{GS}(m) + \mathrm{GS}(n) - 1. \quad [3]$$

That means, for example, that doubling a number makes its *suavitas* one degree lower, i.e., it increases the GS by one, because 2 is prime and therefore GS(2) = 2:

$$\mathrm{GS}(2n) = \mathrm{GS}(n) + \mathrm{GS}(2) - 1 = \mathrm{GS}(n) + 1. \quad [4]$$

Every natural number can be expressed as a unique product of primes. Multiple application of [2] and [3] leads to the general formula for calculation of the GS of any natural number expressed by its prime factors p_1, p_2, ... p_k:

$$\mathrm{GS}(p_1\, p_2 ... p_k) = p_1 + p_2 + ... + p_k - (k - 1). \quad [5]$$

To give a few examples:

$\mathrm{GS}(1) = 1$, $\mathrm{GS}(2) = 2$, $\mathrm{GS}(3) = 3$ (see [1], [2])

$\mathrm{GS}(6) = \mathrm{GS}(3) + 1 = 4$ (see [4])

$\mathrm{GS}(60) = \mathrm{GS}(2{\cdot}2{\cdot}3{\cdot}5) = 2 + 2 + 3 + 5 - 3 = 9$ (see [5])

$\mathrm{GS}(360) = \mathrm{GS}(6) + \mathrm{GS}(60) - 1 = 12$ (see [3])

It is also possible to calculate the last example from the prime factors of 360 using [5]. The result will be the same, of course.

The GS for ratios between two natural numbers is calculated as follows. The ratios $n : 1$ and $1 : n$ have the same simplicity as the number n itself, i.e.,

$$\mathrm{GS}(1 : n) = \mathrm{GS}(n : 1) = \mathrm{GS}(n). \quad [6]$$

Because of $m : n = (m : 1) \cdot (1 : n)$, similar considerations as that which led to [3] give the following (using [6] and [3]):

$$\mathrm{GS}(m : n) = \mathrm{GS}[(m : 1) \cdot (1 : n)] = \mathrm{GS}(m : 1) + \mathrm{GS}(1 : n) - 1 = \mathrm{GS}(m) + \mathrm{GS}(n) - 1$$
$$= \mathrm{GS}(mn),$$

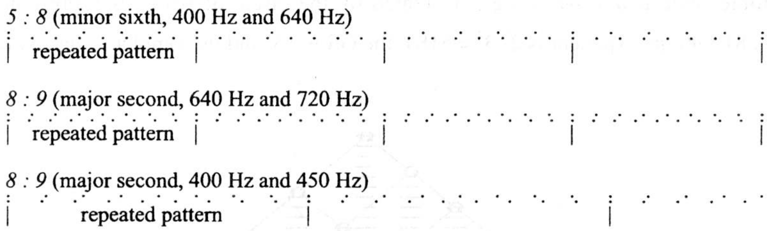
but this is only true if m and n are coprime<5>. Otherwise the ratio must be simplified first, and the *suavitas* will be higher, as, e.g., for GS(6 : 3) = GS(2 : 1) = 2. Thus the general formula for ratios of two natural numbers is:

$$\mathrm{GS}(m : n) = \mathrm{GS}(mn), \text{ if } m \text{ and } n \text{ are coprime,} \quad [7]$$

and the GS(mn) can be calculated using [5].

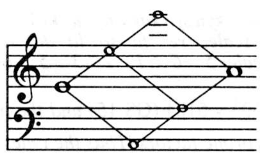
The ratio of the twelfth (octave-and-fifth, 1 : 3), for example, is as simple as that of the interval of two octaves (1 : 4), because GS(3) = GS(4) = 3. The ratio of a minor sixth (5 : 8) is as simple as that of the major second (8 : 9), because GS(5 : 8) = GS(8 : 9) = 8.

The real comprehensibility of these intervals, however, depends on their real frequencies (see statement (3)(*a*) above). While the comprehensibility of a single tone depends on the frequency of the tone itself, it must be assumed that the comprehensibility of a consonant sound depends from the repetition frequency of the vibration pattern that the consonance produces. If, for example, the minor sixth is given by the frequencies 400 Hz and 640 Hz (= 5 : 8), only the major second given by the frequencies 640 Hz and 720 Hz (= 8 : 9) will have the same frequency of repetition. The major second 400 Hz and 450 Hz (= 8 : 9) repeats in a slower pattern and thus is less consonant, as seen in the following graph<6>:



That 5 : 8 and 8 : 9 have the same GS means that the two patterns as such are equally easy to comprehend. But the pattern 8 : 9 contains more vibrations than the pattern 5 : 8. Thus despite having the same GS, a major second is less comprehensible (i.e., less consonant) than a minor sixth, unless it is played with higher frequencies<7>. That intervals become more consonant when they are played with higher pitches can easily be verified on any instrument. Unfortunately Euler does not say *how much* the degree of consonance is increased by a frequency rise<8>.

Musically, the prime factorization of the interval relations can be described as a decomposition of compound intervals into elementary intervals. Only the intervals 1 : p and p : 1 are regarded as elementary. I will therefore call them "prime intervals". The fourth 3 : 4 with GS 5, for example, will be comprehended by going down a twelfth (3:1) and twice going up an octave (1 : 2). These three steps can be done in any order. Thus a fourth will create a system of two explicit and four implicit tones (ex. 2).



Ex. 2. The "tone system" created by a fourth and the tone relations by prime intervals

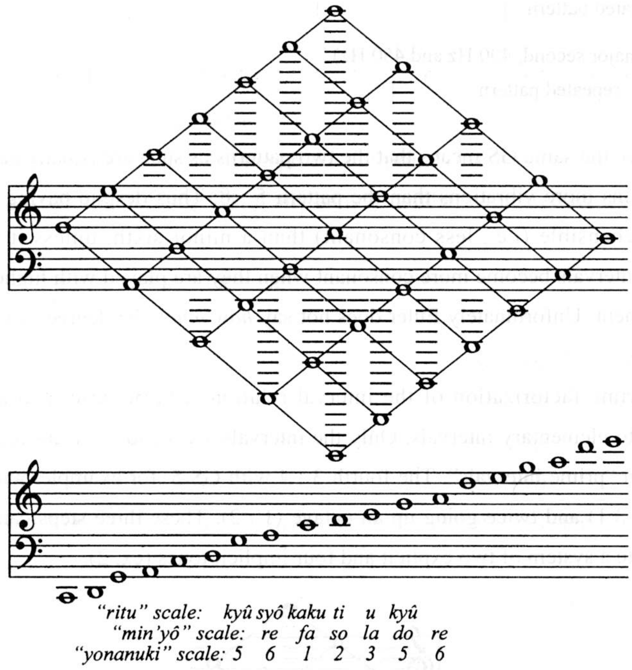
As the four tones A, a, e^2, e^3 are already implied by the fourth e^1–a^1, the system will not become more complex if one or several of the implied tones are added. Thus the *hon-tyôsi* tuning of the *syamisen* e^1–a^1–e^2 and the *niagari-tyôsi* a–e^1–a^1 equally have the GS 5. But again, the consonance of the two tunings is equal only if the second string of the *niagari* is the same pitch as the first string of the *hon-tyôsi*. If, as usual, the tuning of the *syamisen* is changed from *hon-tyôsi* to *niagari* by raising the second string, the consonance of the three strings becomes greater.

If all implicit tones are added to a given consonance, Euler calls it a *complete consonance*. The lowest tone of a complete consonance is represented by the number 1, the highest tone by the *least common multiple* of the other tones, which is called the *exponent* by Euler. If a consonance consists of all divisors of a certain number (including 1 and the number itself), it is a complete consonance, and the certain number is its exponent. The GS of the consonance is given by the GS of its exponent. In ex. 2 the exponent is 12 and the divisors are 1, 2, 3, 4, 6.

So all sets of tones represented by coprime numbers can be enlarged to a complete consonance by adding the least common multiple and all its divisors, and the GS for any set of coprime numbers is given by the GS of their least common multiple.

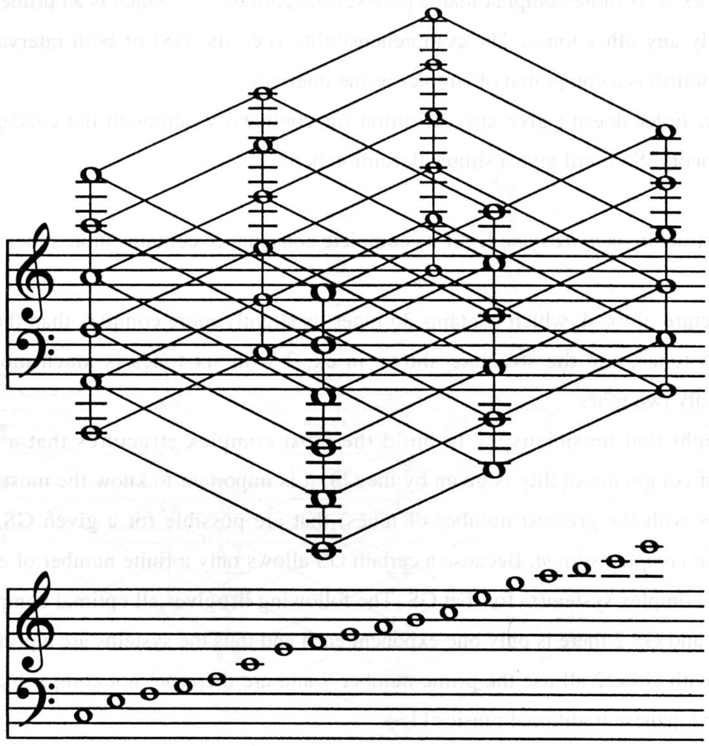
3. The *suavitas* of pentatonic scales

The pentatonic scale g–a–c^1–d^1–e^1–g^1, if created by the circle of fifths, is represented by the ratios 48 : 54 : 64 : 72 : 81 : 96; its exponent is $2^6{\cdot}3^4 = 5184$, the GS is 15, and its complete consonance will appear as shown in ex. 3.



Ex. 3. The complete consonance of exponent $2^6{\cdot}3^4 = 5184$ (GS 15) *and the prime interval relations between its tones. As in ex. 2, the exponent contains two different prime factors. Thus the tones are distributed in a two-dimensional go-board-like space. The tones are spread over a large area, because the highest tone has the 5184-fold frequency of the lowest tone. Both are out of audible range. Only the most middle octave, however, contains the full pentatonic scale. To have the full scale through more octaves, the exponent of the prime-factor* 2 *has to be raised, and the GS will rise accordingly. The lower system shows a section of musically usable tones (see Euler 1739: VIII, 5).*

Although the pentatonic system shown in ex. 3 seems very consonant, even more consonant pentatonic systems can be achieved, if the pure seventeenth (two-octaves-and-third, 1 : 5) is introduced. In fact, already the system with exponent $2^4{\cdot}3^2{\cdot}5 = 720$ and GS 13 contains two pentatonic-like structures, one in the lower range (F–G–A–c–e–f–g–a–c^1–e^1) and one in the upper range (c^1–e^1–g^1–a^1–h^1–c^2–e^2–g^2–a^2–h^2), but they are musically not very usable, because there are two thirds in succession. If, however, a further octave is added, both the so-called "Ryûkyû scale" and the "*miyakobusi*" scale are part of the complete consonance (ex. 4).



"Ryûkyû scale": do mi fa so si do
"miyakubusi" scale: mi fa la si do mi

Ex. 4. The complete consonance of exponent $2^5 \cdot 3^2 \cdot 5 = 1440$ (GS 14). *The tones are distributed in a three-dimensional space because the system comprises three different prime factors.*

From Euler's calculation it follows that *miyakobusi*, the most fundamental pentatonic scale for Japanese traditional music, has the GS 14 and is thus, as a system, more consonant than the pentatonic scale shown in ex. 3. This, however, is true only if we consider the major thirds *c–e* and *f–a* to be pure thirds (4 : 5), at least in principle. This seems to contradict the theory of *gagaku* but is in accordance with the measurements of Uehara Rokusirô for the *miyakobusi* scale<9>. In fact, it is well known that the theory of *gagaku* does not reflect the actual consonances used in that music<10>. And that the thirds are pure intervals "in principle" does not necessarily mean that they require just intonation. (Also in modern Western music triads are seldom played with pure intonation, but most theorists believe that the frequency relation of 4 : 5 : 6 is constitutive for the major triad.) Euler's theory thus is a strong argument for the opinion that the major thirds in both the *miyakobusi* and the Ryûkyû scale are recognized as consonant intervals by Japanese listeners.

4. Comprehensibility and complexity

Euler assumes that people like trying to comprehend complex structures (statement (2) above). Comparing ex. 3 and ex. 4, however, it seems that "more complex" is not synonymous with "less comprehensible": although the complete consonance in ex. 3 is less comprehensible, because it has a lower *suavitas*, it seems to be not more complex as a structure. To give a more striking example: Certainly a fourth, which implies the structure shown in ex. 2, is more complex than a pure seventeenth (C–e^1), which is an prime interval (1 : 5) and thus does not imply any other tones. The comprehensibility (i.e., the GS) of both intervals, however, is the same, because the fourth is a compound of simpler prime intervals.

Unfortunately, Euler doesn't give any definition for complexity, although the concept of complexity is important for his theory. So I will give a simple definition here:

Def.: A structure is *more complex* if its complete consonance contains more tones.

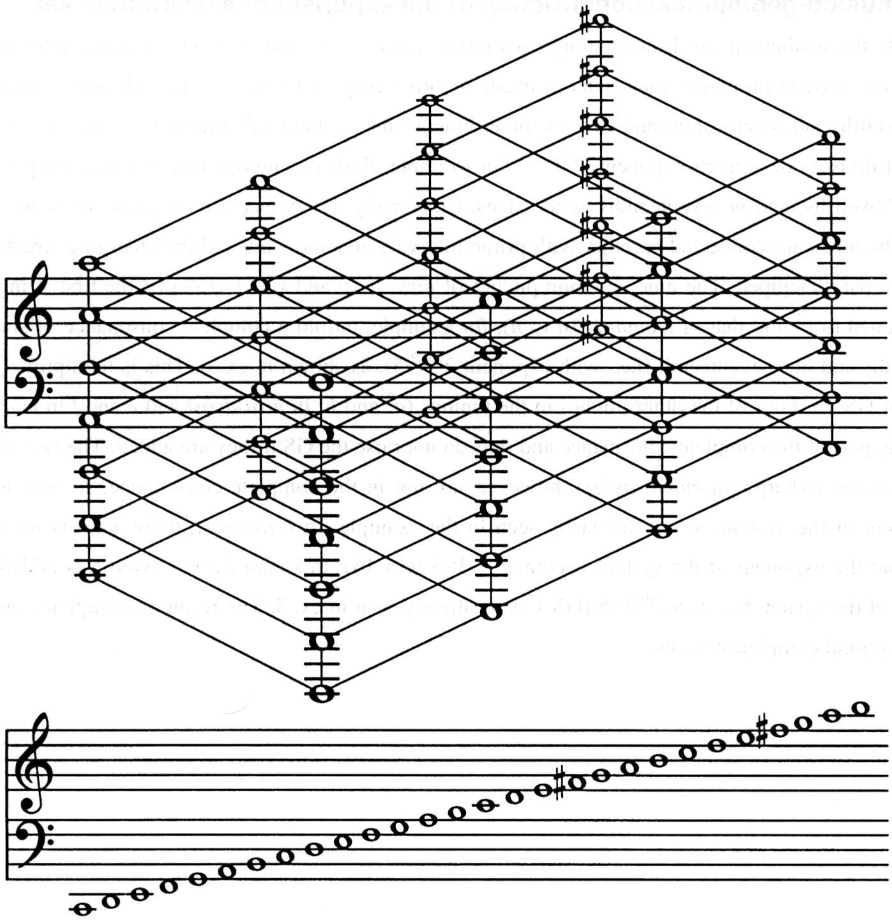
Thus the structure of ex. 4, which contains 36 tones, is slightly more complex than the structure of ex. 3, which contains 35 tones, and the structure shown in ex. 2 with six tones is much more complex than a seventeenth with only two tones.

If Euler is right that musicians try to build the most complex structures that a listener is able to comprehend, and if comprehensibility is given by the GS, it is important to know the most complex structures (i.e., the structures with the greatest number of tones) that are possible for a given GS. I will call such a structure an *optimal complex system*. Because a certain GS allows only a finite number of exponents, it is easy to find the optimal complex system(s) for that GS. The following list gives all optimal complex systems for GS 3 to 22. (For GS 1 and GS 2 there is only one exponent each and thus the systems are a priori optimal. Optimal complex systems with GS>22 all use the prime number 7 and are therefore not compatible with tone systems used in Western or Japanese traditional music<11>).

gradus suavitatis (GS)	exponent	number of tones
3	$2^2 = 4$	3
4	$2 \cdot 3 = 6$ *and* $2^3 = 8$	4
5	$2^2 \cdot 3 = 12$	6
6	$2^3 \cdot 3 = 24$	8
7	$2^4 \cdot 3 = 48$	10
8	$2^3 \cdot 3^2 = 72$ *and* $2^5 \cdot 3 = 96$	12
9	$2^4 \cdot 3^2 = 144$	15
10	$2^5 \cdot 3^2 = 288$	18
11	$2^6 \cdot 3^2 = 576$	21
12	$2^3 \cdot 3^2 \cdot 5 = 360$ *and* $2^5 \cdot 3 \cdot 5 = 480$ *and* $2^5 \cdot 3^3 = 864$ *and* $2^7 \cdot 3^2 = 1152$	24
13	$2^4 \cdot 3^2 \cdot 5 = 720$	30
14	$2^5 \cdot 3^2 \cdot 5 = 1440$	36
15	$2^6 \cdot 3^2 \cdot 5 = 2880$	42
16	$2^5 \cdot 3^3 \cdot 5 = 4320$ *and* $2^7 \cdot 3^2 \cdot 5 = 5760$	48
17	$2^6 \cdot 3^3 \cdot 5 = 8640$	56
18	$2^7 \cdot 3^3 \cdot 5 = 17280$	64
19	$2^8 \cdot 3^3 \cdot 5 = 34560$	72
20	$2^7 \cdot 3^4 \cdot 5 = 51840$ *and* $2^9 \cdot 3^3 \cdot 5 = 69120$	80
21	$2^8 \cdot 3^4 \cdot 5 = 103680$	90
22	$2^9 \cdot 3^4 \cdot 5 = 207360$	100

Optimal complex systems for GS 3 to 22

Clearly the systems in ex. 2 and ex. 4 are optimal complex systems, but not the system in ex. 3. For use as a musical scale the optimal complex system for GS 17 is particularly interesting, because it contains a diatonic scale through the full range of human voices (ex. 5).



Ex. 5. The complete consonance of exponent $2^6{\cdot}3^3{\cdot}5 = 8640$ (GS 17). *It is one of the optimal complex systems.*

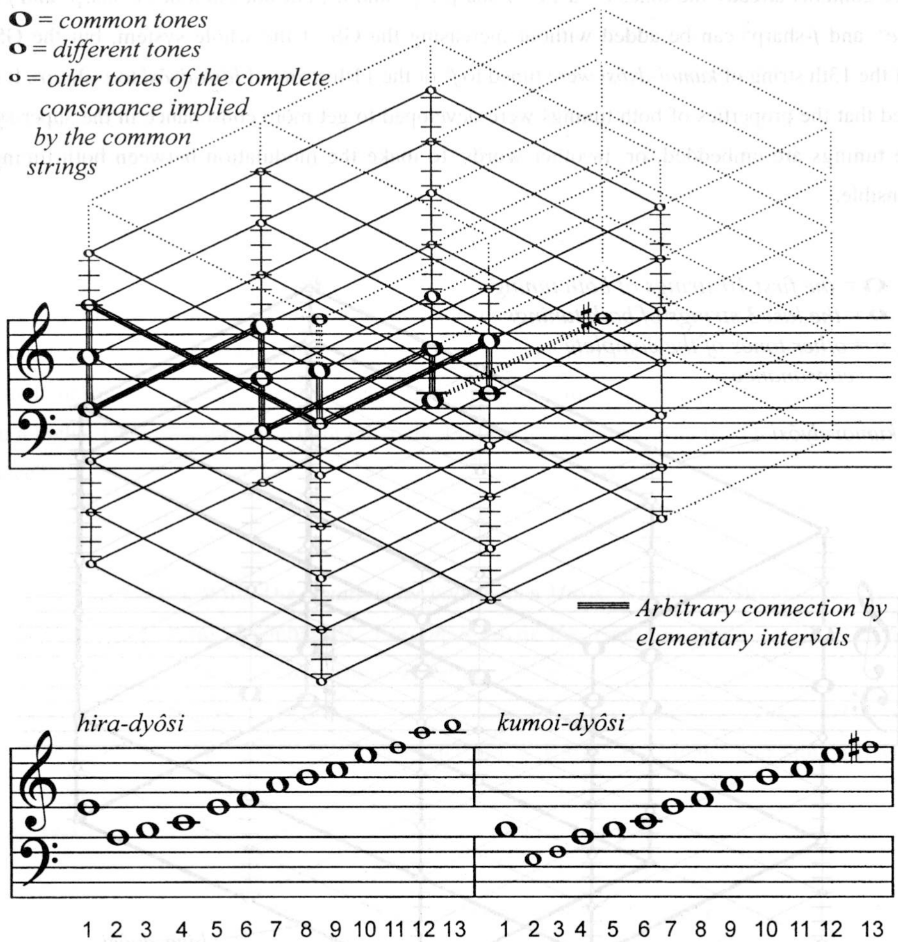
In the lower two octaves the scale has only natural notes, but in the upper two octaves it has *f*-sharp instead of *f*. If transposed one octave upwards, the scale contains all notes shown in the scales of ex. 1. In fact, all scales given in ex. 1 have the same exponent, $2^6{\cdot}3^3{\cdot}5 = 8640$, and therefore the GS 17. From the perspective of Euler's theory, therefore, the only difference between the four scales is which tones are used and all represent the same complete consonance. It even makes no difference whether the real scale is pentatonic, as in ex. 1 (b), or heptatonic, as in exx. 1 (a), (b) and (d), because all other notes are part of the complete consonance.

So, astonishingly enough, traditional Japanese musicians as well as medieval Western composers have invented the most perfect diatonic scale possible. Why did later Western composers give up that system? Euler recommends building instruments that can play all scale notes in all octaves for practical reasons (Euler 1739: VIII, 5), although he does not think that it is good to use all these notes without distinction in a composition.

But transposability, use of modulations and singing in octaves may have been the reasons that the concept of consonance was partly given up in Western music.

5. The musico-geometrical construction of the exponent of a given tone set

While the mathematician Euler mainly uses ratios, calculations and concepts of number theory to find solutions for musical problems, musicians or music theorists may prefer to operate with prime intervals (i.e., octave, twelfth, and seventeenth) and two- or three-dimensional geometrical structures as shown in exx. 2 - 5. The set of divisors of a given exponent $2^k \cdot 3^l \cdot 5^m$, for example, finds its embodiment in a parallelepiped with k octaves, l twelfths, and m seventeenths as its edges. Conversely the exponent of a given set of tones can be found if the tones are connected together with prime intervals in some way, and the emerging structure is enlarged to a parallelepiped. The nine common pitches of *hira-dyôsi* and *kumoi-dyôsi* (if the first string of *hira-dyôsi* is tuned to e^1 and that of *kumoi-dyôsi* to h), for example, unfold a complete consonance of six octaves, two twelfths and one seventeenth (i.e., with exponent $2^6 \cdot 3^2 \cdot 5$), as shown in ex. 6. This is the optimal complex system for GS 15. Four of the other pitches in the tunings (a^2 and h^2 in *hira-dyôsi* and *e* and *f* in *kumoi-dyôsi*) are already part of the complete consonance and thus do not raise the GS if they are added. The two notes f^2 (in *hira-dyôsi*) and *f*-sharp^2 (in *kumoi-dyôsi*), however, are not in the complete consonance, so they lead to an enlargement of the system, as can be easily seen in the example. In *hira-dyôsi*, there appears an additional octave, and the exponent of the system becomes $2^7 \cdot 3^2 \cdot 5$ (GS 16). In *kumoi-dyôsi* a twelfth is added, and the exponent of the system becomes $2^6 \cdot 3^3 \cdot 5$ (GS 17), as already seen in ex. 5. The resulting complete consonances are both optimal complex systems.

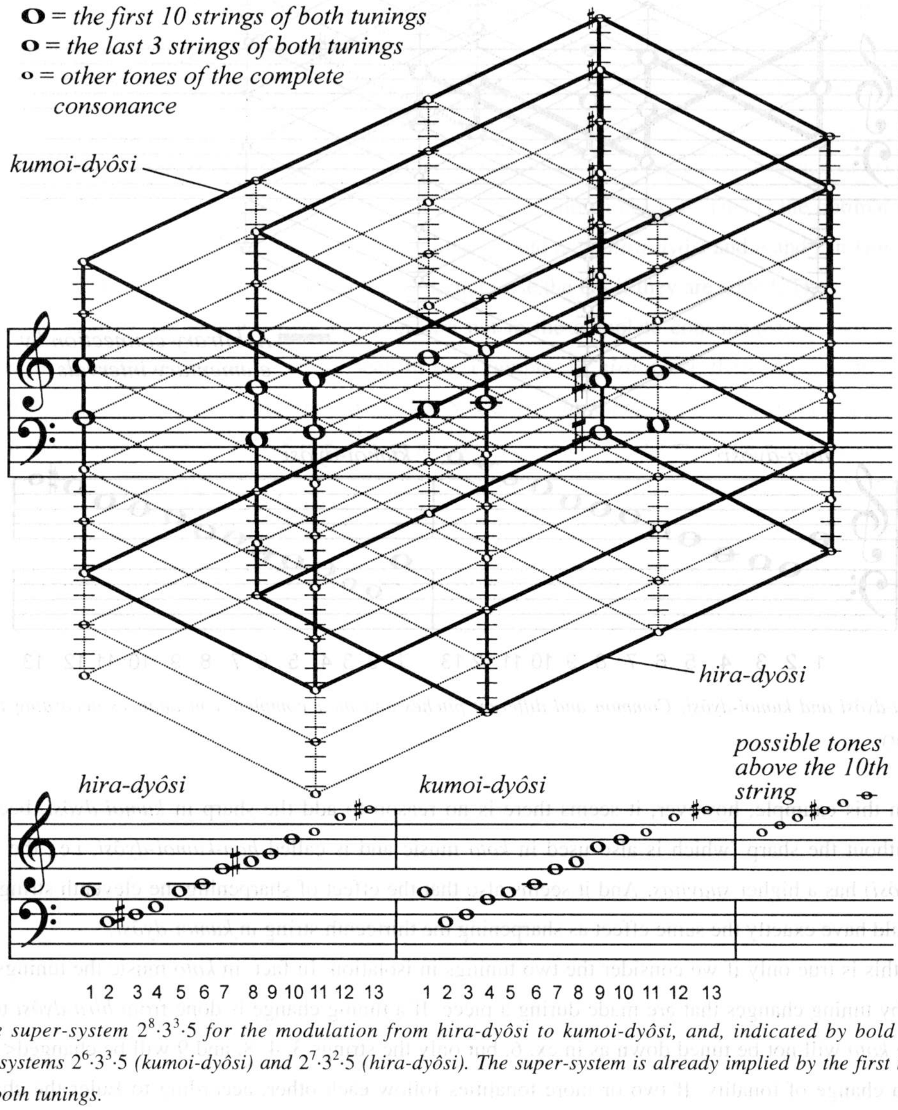


Ex. 6: Hira-dyôsi and kumoi-dyôsi. Common and different pitches and their complete consonances according to Euler's theory

From this example, however, it seems there is no reason to add the sharp in *kumoi-dyôsi*, because the tuning without the sharp (which is also used in *koto* music and is called *hon-kumoi-dyôsi*, i.e., the "proper" *kumoi-dyôsi*) has a higher *suavitas*. And it seems also that the effect of sharpening the eleventh string of *hira-dyôsi* would have exactly the same effect as sharpening the thirteenth string in *kumoi-dyôsi*.

But this is true only if we consider the two tunings in isolation. In fact, in *koto* music the tunings are tied together by tuning changes that are made during a piece. If a tuning change is done from *hira-dyôsi* to *kumoi-dyôsi*, the *koto* will not be tuned down as in ex. 6, but only the strings 3, 4, 8, and 9 will be changed<12>. The result is a change of tonality. If two or more tonalities follow each other, according to Euler the change can only be comprehended if the systems are understood as subsets of a super-system. Therefore we have to consider the complete consonance that includes both tunings.

It is our aim to discuss the tuning of the uppermost three strings in both tunings. Hence, as the first step, we will consider the super-system that is unfolded by the first ten strings of both tunings. The system has the exponent $2^8 \cdot 3^3 \cdot 5$ and represents the optimal complex system for GS 19. As ex. 7 shows, the complete consonance contains already the tones c^2, d^2, e^2, *f*-sharp^2, g^2, and a^2, but not the tones *c*-sharp^2 and f^2. So the tones c^2, e^2, and *f*-sharp^2 can be added without increasing the GS of the whole system, but the GS would increase if the 13th string of *kumoi-dyôsi* were tuned to f^2 or the 11th string of *hira-dyôsi* to *c*-sharp^2. It can thus be assumed that the properties of both tunings were developed to get more consonance in the super-system in which the tunings are embedded, or, in other words, to make the modulation between both tunings more comprehensible.



Ex. 7: The super-system $2^8{\cdot}3^3{\cdot}5$ for the modulation from hira-dyôsi to kumoi-dyôsi, and, indicated by bold lines, the subsystems $2^6{\cdot}3^3{\cdot}5$ (kumoi-dyôsi) and $2^7{\cdot}3^2{\cdot}5$ (hira-dyôsi). The super-system is already implied by the first ten strings of both tunings.

6. Conclusion

The concept of *optimal complex systems* introduced in this paper according to the principles of Euler's music theory is particularly useful to explain the properties of Japanese tunings and scales. It was shown at several levels that Japanese tone systems achieve the greatest possible consonance as defined by "the highest possible complexity at a certain grade of comprehensibility". The *hon-tyôsi* and *niagari-tyôsi* of the *syamisen*, the *hira-dyôsi* and *kumoi-dyôsi* of the *koto*, and the super-system that contains the modulation from *hira-dyôsi* to *kumoi-dyôsi* all represent optimal complex systems<13>. It would be easy to demonstrate that the tunings of violins, lutes, guitars, pianos and so on do *not* represent such systems.

Maybe it sounds strange that a European music theorist of the 18th century, who did not know any Japanese music, was able to find fundamental principles that are more suitable to explain Japanese than Western music. It was possible, however, because his music theory was based not on musical experience but on a theory of comprehension that is based on elementary mathematical principles.

Certainly music cannot be reduced to the question of comprehension and solving puzzles. But comprehensibility is an important aspect of every form of communication. Why should the Japanese, who refined their music over several hundred years in a steady tradition of listening and performing, not have achieved a higher level of comprehensibility than the European composers, who tried to make a new theory for every piece?

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Notes

1 See Apel 1981: 108. Examples for this practice are found in the same book in facsimiles 23, 28, 31, 32, 39, 41, 66, 72, 82, 84, 85, 86.

2 Euler 1739. Because there are various editions and translations, for the citations the chapter and paragraph numbers are used. For this paper a copy of the original edition was used. For an overview and bibliography on Euler as music theorist see Gottschewski 2001.

3 In fact, Euler's theory of consonance is not restricted to tone relations but may also explain rhythmical and visual phenomena. (At least Euler believed so.)

4 According to Euler, irrational frequency relations are not perceptible at all. If tempered or detuned intervals are recognized as something, it is through adaptive listening: The intervals are perceived as in a rational relation that is near to the irrational relation presented.

5 *Coprime* means that the greatest common divisor of two or more numbers is 1.

6 A similar graphic representation is used by Euler in a tabula to cap. II, § 21.

7 This is a compelling conclusion from Euler's fundamental statements, but Euler did not draw it. (Rather he

tries to find an explanation for the difference in consonance in compositional practice; see cap. IV, § 14 - 15.) Thus it was frequently overlooked, and GS was identified with consonance. So it was wrongly criticized that Euler's theory leads to the consequence that the major triad (4 : 5 : 6) and the minor triad (10 : 12 : 15) are equally consonant. This is only the case if the real frequency ratio between the roots of the two triads is 4 : 10. It follows that the minor triad is less consonant if the two chords are played in the same pitch range.

8 I don't see that there is any statement in Euler's theory that could lead us to a measure for that, but at least it can be said that the rise of one octave increases the consonance by not more than one degree, because otherwise an octave would be easier to comprehend than its upper tone alone.

9 See Uehara 1895, chapter 23.

10 In theoretical writings on *gagaku* a circle of pure fifths is described that begins ascending from *d* (*itikotu, ichikotsu*) and ends with *g* (*sôdyô, sojô*). Thus theoretically *g–d* cannot represent a consonant interval. In practice, however, it is a fundamental consonance in modes like *sôdyô* and in some instrumental tunings.

11 Euler proposed an enlargement of our 12-tone system by 12 additional tones to make the use of the natural seventh possible, but only a few composers in the 18th and 20th centuries tried to compose pieces with such a tone system. See Gottschewski 2001.

12 Normally the change will go through another tuning as *kata-kumoi-dyôsi* or *han-kumoi-dyôsi*, which are called "half" *kumoi*, because some of the strings are as in *hira-dyôsi* and others are as in *kumoi-dyôsi*.

13 Also many of the other tunings and modulations of Japanese instruments unfold optimal complex systems: The most important tunings and modulations of the *syamisen*, namely *sansagari* ($2^4 \cdot 3^2$, GS 9; this system already implies the possibility of retuning the first string down to *niagari* or the third string up to *hon-tyôsi*), the modulation from *hon-tyôsi* to *niagari* by raising the second string ($2^3 \cdot 3^2$, GS 8), and the modulation row *hon-tyôsi*, *niagari*, *sansagari* by subsequently raising the second and first strings ($2^5 \cdot 3^3$, GS 12); almost all *koto* tunings, including *nizyû-kumoi-dyôsi* and *nakazora-dyôsi* (both as *hira-dyô*si $2^7 \cdot 3^2 \cdot 5$, GS 16), *kokin-dyôsi* ($2^7 \cdot 3^3 \cdot 5$, GS 18), and *akebono-dyôsi* ($2^7 \cdot 3^4 \cdot 5$, GS 20); and most *biwa* tunings, as for example four of the six tunings of the *gaku-biwa: itikotu-tyô* ($2^3 \cdot 3^2$, GS 8), *hyôdyô*, *sôdyô*, and *suityô* (all $2^4 \cdot 3^2$, GS 9).